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Scalar metric fluctuations in space–time matter inflation

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Abstract

Using the Ponce de León background metric, which describes a 5D universe in an apparent vacuum: $\bar{G}_{AB} = 0$, we study the effective 4D evolution of both, the inflaton and gauge-invariant scalar metric fluctuations, in the recently introduced model of space–time matter inflation.

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1. Introduction and motivation

The idea that the universe passed through an inflationary expansion in early epochs has become an integral part of the standard cosmological model. Postulating a period of nearly exponential growth in the primordial universe, inflationary cosmology solved many problems which plagued previous models of the big bang and also provides a mechanism for the creation of primordial density fluctuations. During inflation, the universe is dominated by the inflaton, the scalar field whose evolution controls the dynamics of the expansion. The first model of inflation was proposed by Starobinsky in 1979 [1]. Two years later, a model with a clear motivation was developed by Guth [2], in order to solve some of the shortcomings of the big bang theory, and in particular, to explain the extraordinary homogeneity of the observable universe. The scalar field state employed in the original version of inflation is called a false vacuum, since the state temporarily acts as if it were the state of lowest possible energy density. In 1983 Linde proposed chaotic inflation [3]. In its standard version, the inflaton field is related to a potential with a local minimum or a gently plateau. In this model, cosmological perturbations are indispensable in relating early universe scenarios [4]. Other standard 4D models where dissipative effects are important during the inflationary phase, were suggested more recently [5–7]. The evolution of gauge-invariant metric perturbations during inflation has been well studied [8]. This allows to formulate the problem of the amplitude for the scalar metric perturbations on the evolution of the background Friedmann–Robertson–Walker (FRW) universe in a coordinate-independent manner at every moment in time.

On the other hand, cosmological theories with extra dimensions are already known to be of great importance in cosmology [9,10]. During the last years there were many attempts to construct a consistent brane world (BW) cosmology [11]. The induced-matter, or space–time-matter (STM) theory stands out for their closeness to the Einstein's project of considering matter and radiation as manifestations of pure geometry [12]. The basic extension of 4D Einstein theory and the low-energy limit of higher-dimensional theories is the modern incarnation of noncompact 5D Kaluza–Klein theory. In the STM theory, the conjecture is that ordinary matter and fields that we observe in 4D are induced geometrically by the extra dimension [13]. In this Letter we develop a consistent first-order formalism to study the inflaton and scalar metric fluctuations [14] in STM inflation, which was recently introduced [15].

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2. 5D formalism

We consider a 5D background metric, which is 3D spatially isotropic, homogeneous and flat. In this Letter we shall consider the background metric: $dS^2_{(b)} = \bar{g}_{AB}\,dx^A\,dx^B$,[1] introduced by Ponce de León [9]

$$dS^2_{(b)} = l^2\,dt^2 - \left(\frac{t}{t_0}\right)^{2p} l^{\frac{2p}{p-1}}\,dr^2 - \frac{t^2}{(p-1)^2}\,dl^2, \tag{1}$$

where $dr^2 = dx^2 + dy^2 + dz^2$ and l is the fifth coordinate (which is space-like). The background metric (1) represents a 5D apparent vacuum $\bar{G}_{AB} = 0$, but however it is no flat. The absolute value for the determinant of the background metric tensor $\bar{g}_{AB}$ is

$$\left|{}^{(5)}\bar{g}\right| = \left[\frac{t^{3p+1}\,l^{\frac{4p-1}{p-1}}}{(p-1)t_0^{3p}}\right]^2.$$

To describe the system, we consider the action

$$I = -\int d^4x\,dl\,\sqrt{\frac{{}^{(5)}\bar{g}}{{}^{(5)}\bar{g}_0}}\left[\frac{{}^{(5)}\bar{R}}{16\pi G} + {}^{(5)}\mathcal{L}\right], \tag{2}$$

where ${}^{(5)}\bar{R} = 0$ is the 5D background Ricci scalar for the background metric (1) and $G = M_p^{-2}$ is the Newton's constant ($M_p = 1.2 \times 10^{19}$ GeV is the Planckian mass). We shall use the unities $c = \hbar = 1$, being c and h the speed of light and the Planck's constant. The Lagrangian density in (2) is given by

$${}^{(5)}\mathcal{L} = \frac{1}{2} g^{AB}\varphi_{,A}\varphi_{,B}, \tag{3}$$

which is only kinetic because we are dealing with a 5D free scalar field φ in an apparent vacuum state. Furthermore, $g^{AB} = \bar{g}^{AB} + \delta g^{AB}$ is the perturbed contravariant metric tensor and $\bar{g}^{AB}$ is the background contravariant metric tensor. In this Letter we shall consider δg^{AB} as the scalar perturbations of the metric at first order, because we are dealing with weak gravitational fields. The diagonal perturbed metric (with respect to the background metric (1)), is

$$dS^2 = l^2(1+2\Phi)\,dt^2 - \left(\frac{t}{t_0}\right)^{2p} l^{\frac{2p}{p-1}}(1-2\Psi)\,dr^2 - \frac{t^2}{(p-1)^2}(1-Q)\,dl^2, \tag{4}$$

where $\Phi(t,\vec{r},l)$, $\Psi(t,\vec{r},l)$ and $Q(t,\vec{r},l)$ are the scalar metric fluctuations. In particular for $\Phi = \Psi$ and $Q = 2\Psi$, we obtain the following line element:

$$dS^2 = l^2(1+2\Psi)\,dt^2 - \left(\frac{t}{t_0}\right)^{2p} l^{\frac{2p}{p-1}}(1-2\Psi)\,dr^2 - \frac{t^2}{(p-1)^2}(1-2\Psi)\,dl^2. \tag{5}$$

The contravariant metric tensor, after make a Ψ-first-order approximation, is given by

$$\begin{aligned} g^{AB} = \mathrm{diag}\Bigg[&(1-2\Psi)/l^2, -(1+2\Psi)l^{\frac{-2p}{(p-1)}}\left(\frac{t}{t_0}\right)^{-2p}, -(1+2\Psi)l^{\frac{-2p}{(p-1)}}\left(\frac{t}{t_0}\right)^{-2p}, \\ &-(1+2\Psi)l^{\frac{-2p}{(p-1)}}\left(\frac{t}{t_0}\right)^{-2p}, -(1+2\Psi)\frac{(p-1)^2}{t^2}\Bigg], \end{aligned} \tag{6}$$

which can be written as $g^{AB} = \bar{g}^{AB} + \delta g^{AB}$, being $\bar{g}^{AB}$ the contravariant background metric tensor. Furthermore, the Lagrangian ${}^{(5)}\mathcal{L} = \frac{1}{2}g^{AB}\varphi_{,A}\varphi_{,B}$, with the metric fluctuations Ψ included in g^{AB}, now can be written as

$${}^{(5)}\mathcal{L}(\varphi, \psi, \varphi_{,A}, \psi_{,A}) = \frac{1}{2} g^{AB}\varphi_{,A}\varphi_{,B},$$

where the fields φ and Ψ play the role of coordinates and g^{AB} is given by (6).

The relevant components of the linearized Einstein tensor are

$$G_{tt} = \frac{1}{t^2}\left[-3l^2(p-1)^2\overset{\star\star}{\Psi} + 9l(p-p^2)\overset{\star}{\Psi} + 3t(3p+1)\dot{\Psi} + 12(p^2+p)\Psi - 3t^2\left(\frac{t_0}{t}\right)^{2p} l^{\frac{-2}{p-1}}\nabla_r^2\Psi\right], \tag{7}$$

[1] In our conventions, capital Latin indices run from 0 to 4.

$$G_{rr} = 3\left(\frac{t}{t_0}\right)^{2p} \frac{l^{\frac{2p}{p-1}}}{t^2 l^2}\left[l^2(p-1)^2 \overset{\star\star}{\Psi} + l\left(p^2-1\right)\overset{\star}{\Psi} - 3t^2\ddot{\Psi} - 5t(2p+1)\dot{\Psi} - 12p^2\Psi\right] + 2\nabla_r^2\Psi, \tag{8}$$

$$G_{ll} = \frac{1}{l^2(p-1)^2}\left[3l\left(2p^2-3p+1\right)\overset{\star}{\Psi} - 3t^2\ddot{\Psi} - 15pt\dot{\Psi} + 12p(1-2p)\Psi + \left(\frac{t_0}{t}\right)^{2p} l^{\frac{-2}{p-1}} t^2 \nabla_r^2\Psi\right], \tag{9}$$

$$G_{tl} = -3\left[\frac{1}{2}\dot{\overset{\star}{\Psi}} + \frac{(2p-1)}{t}\overset{\star}{\Psi} + \frac{1}{l}\dot{\Psi}\right], \tag{10}$$

where the overstar denotes the derivative with respect to the fifth coordinate l. On the other hand, the energy–momentum tensor can also be written as a linear one, with background components $\bar{T}_{AB}$, plus the first order perturbations δT_{AB}: $T_{AB} = \bar{T}_{AB} + \delta T_{AB} = \varphi_{,A}\varphi_{,B} - g_{AB}\varphi_{,C}\varphi^{,C}$, where

$$\bar{T}_{AB} = \varphi_{,A}\varphi_{,B} - \bar{g}_{AB}\varphi_{,C}\varphi^{,C}\big|_{\text{background}} = 0, \tag{11}$$

$$\delta T_{AB} = -\delta g_{AB}\varphi_{,C}\varphi^{,C}. \tag{12}$$

Notice that all the terms δT_{AB} (with $A \neq B$) are zero, because δg_{AB} is diagonal. The relevant perturbed components are

$$\delta T_{tt} = 2\Psi(p-1)^2\frac{l^2}{t^2}\left(\overset{\star}{\varphi}\right)^2, \tag{13}$$

$$\delta T_{rr} = -\frac{6\Psi}{l^2}\left(\frac{t}{t_0}\right)^{2p} l^{\frac{2p}{p-1}}(\dot{\varphi})^2, \tag{14}$$

$$\delta T_{ll} = -\frac{2\Psi t^2}{(p-1)^2 l^2}(\dot{\varphi})^2, \tag{15}$$

which are nonlinear. In order to solve the perturbed and background Einstein equations, we consider the following semiclassical expansion for the inflaton field: $\varphi(t,\vec{r},l) = \varphi_{(b)}(t,l) + \delta\varphi(t,\vec{r},l)$, being $\varphi_{(b)}$ the background inflaton field and $\delta\varphi$ the quantum fluctuations such that $\langle\delta\varphi\rangle = 0$. If we take into account this semiclassical expansion in Eqs. (13)–(15), the linearized Einstein equations $\delta G_{AB} = 8\pi G\delta T_{AB}$ (such that $G_{AB} = \bar{G}_{AB} + \delta G_{AB}$ with $\bar{G}_{AB} = 0$), hold

$$\begin{aligned}&-3l^2(p-1)^2\overset{\star\star}{\Psi} + 9l\left(p-p^2\right)\overset{\star}{\Psi} + 3t(3p+1)\dot{\Psi} + 12\left(p^2+p\right)\Psi - 3t^2\left(\frac{t_0}{t}\right)^{2p} l^{\frac{-2}{p-1}}\nabla_r^2\Psi \\ &\quad = -16\pi G\Psi(p-1)^2 l^2\left(\overset{\star}{\varphi}_{(b)}\right)^2,\end{aligned} \tag{16}$$

$$l^2(p-1)^2\overset{\star\star}{\Psi} + l\left(p^2-1\right)\overset{\star}{\Psi} - 3t^2\ddot{\Psi} - 5t(2p+1)\dot{\Psi} - 12p^2\Psi + \frac{2}{3}t^2\left(\frac{t_0}{t}\right)^{2p} l^{\frac{-2}{p-1}}\nabla_r^2\Psi = 16\pi G\Psi t^2(\dot{\varphi}_{(b)})^2, \tag{17}$$

$$3l\left(2p^2-3p+1\right)\overset{\star}{\Psi} - 3t^2\ddot{\Psi} - 15pt\dot{\Psi} + 12p(1-2p)\Psi + t^2\left(\frac{t_0}{t}\right)^{2p} l^{\frac{-2}{p-1}}\nabla_r^2\Psi = 16\pi G\Psi t^2(\dot{\varphi}_{(b)})^2. \tag{18}$$

Combining Eqs. (16)–(18), we obtain an equation of motion for Ψ

$$\begin{aligned}&t^2\ddot{\Psi} - 2(p-1)^2 l^2\overset{\star\star}{\Psi} + 3(p+2)t\dot{\Psi} + 3(1-p)l\overset{\star}{\Psi} + 12p\Psi - t^2\left(\frac{t_0}{t}\right)^{2p} l^{\frac{-2}{p-1}}\nabla_r^2\Psi \\ &\quad = -\frac{16}{3}\pi G\Psi\left[t^2(\dot{\varphi}_{(b)})^2 + l^2(p-1)^2\left(\overset{\star}{\varphi}_{(b)}\right)^2\right].\end{aligned} \tag{19}$$

The Lagrange equation for Ψ: $\frac{\partial^{(5)}L}{\partial\Psi} - \frac{\partial}{\partial x^A}\left(\frac{\partial^{(5)}L}{\partial\Psi_{,A}}\right) = 0$, is given by

$$\frac{(\dot{\varphi})^2}{l^2} + \left(\frac{t}{t_0}\right)^{2p} l^{\frac{2p}{p-1}}(\nabla_r\varphi)^2 + \frac{(p-1)^2}{t^2}\left(\overset{\star}{\varphi}\right)^2 = 0, \tag{20}$$

where ${}^{(5)}L = \sqrt{\left|\frac{{}^{(5)}g}{{}^{(5)}g_0}\right|}\,{}^{(5)}\mathcal{L}$. In absence of the fluctuations $\delta\varphi$ and Ψ, Eq. (20) gives us

$$\frac{(\dot{\varphi}_{(b)})^2}{l^2} + \frac{(p-1)^2}{t^2}\left(\overset{\star}{\varphi}_{(b)}\right)^2 = 0. \tag{21}$$

Furthermore, the Lagrange equation for φ: $\frac{\partial^{(5)}L}{\partial\varphi} - \frac{\partial}{\partial x^A}\left(\frac{\partial^{(5)}L}{\partial\varphi_{,A}}\right) = 0$, is

$$(3p+1)\dot{\varphi}+t\ddot{\varphi}-\frac{l^{\frac{-2}{p-1}}t_0^{2p}}{t^{2p-1}}\nabla_r^2\varphi-(p-1)^2\frac{l}{t}\left[\frac{4p-1}{p-1}\overset{\star}{\varphi}+l\overset{\star\star}{\varphi}\right]$$
$$-2\Psi\left\{(3p+1)\dot{\varphi}+t\ddot{\varphi}+\frac{l^{\frac{-2}{p-1}}t_0^{2p}}{t^{2p-1}}\nabla_r^2\varphi+(p-1)^2\frac{l}{t}\left[\frac{4p-1}{p-1}\overset{\star}{\varphi}+l\overset{\star\star}{\varphi}\right]\right\}$$
$$-2\left\{t\dot{\Psi}\dot{\varphi}+\frac{l^{\frac{-2}{p-1}}t_0^{2p}}{t^{2p-1}}\left[\frac{\partial\Psi}{\partial x}\frac{\partial\varphi}{\partial x}+\frac{\partial\Psi}{\partial y}\frac{\partial\varphi}{\partial y}+\frac{\partial\Psi}{\partial z}\frac{\partial\varphi}{\partial z}\right]+(p-1)^2\frac{l^2}{t}\overset{\star}{\Psi}\overset{\star}{\varphi}\right\}=0. \tag{22}$$

Notice that Eqs. (20) and (22) are very difficult to be solved because φ and Ψ are quantum operators. However, Eq. (20) can be treated on the background (see Eq. (21)) and Eq. (22) can be linearized using the semiclassical expansion $\varphi(t,\vec{r},l)=\varphi_{(b)}(t,l)+\delta\varphi(t,\vec{r},l)$. The linearized Lagrangian equations take the form

$$\ddot{\varphi}_{(b)}+\left(\frac{3p+1}{t}\right)\dot{\varphi}_{(b)}-\frac{(p-1)^2l^2}{t^2}\overset{\star\star}{\varphi}_{(b)}-(p-1)(4p-1)\frac{l}{t^2}\overset{\star}{\varphi}_{(b)}=0, \tag{23}$$

$$\ddot{\delta\varphi}+\frac{3p+1}{t}\dot{\delta\varphi}-\frac{(p-1)^2l^2}{t^2}\overset{\star\star}{\delta\varphi}-(p-1)(4p-1)\frac{l}{t^2}\overset{\star}{\delta\varphi}-l^{\frac{-2}{p-1}}\left(\frac{t_0}{t}\right)^{2p}\nabla_r^2\delta\varphi$$
$$=2\Psi\left[\ddot{\varphi}_{(b)}+\frac{3p+1}{t}\dot{\varphi}_{(b)}+\frac{(p-1)^2l^2}{t^2}\overset{\star\star}{\varphi}_{(b)}+(p-1)(4p-1)\frac{l}{t^2}\overset{\star}{\varphi}_{(b)}\right]+2\left[\dot{\Psi}\dot{\varphi}_{(b)}+(p-1)^2\frac{l^2}{t^2}\overset{\star}{\Psi}\overset{\star}{\varphi}_{(b)}\right]. \tag{24}$$

The only solution of Eq. (23) on the manifold (1) that is solution of Eq. (21) is $\varphi_{(b)}(t,l)=C$, where C is a constant. Hence, the right side of Eq. (24) becomes zero. If we take into account the expression (21), Eq. (19) holds

$$\ddot{\Psi}-2(p-1)^2\frac{l^2}{t^2}\overset{\star\star}{\Psi}+\frac{3(p+2)}{t}\dot{\Psi}+3(1-p)\frac{l}{t^2}\overset{\star}{\Psi}+12\frac{p}{t^2}\Psi-\left(\frac{t_0}{t}\right)^{2p}l^{\frac{-2}{p-1}}\nabla_r^2\Psi=0. \tag{25}$$

In order to simplify the structure of this equation we propose the transformation $\Psi(t,\vec{r},l)=(\frac{t}{t_0})^{-\frac{3(p+2)}{2}}(\frac{l}{l_0})^{-\frac{3}{4(p-1)}}\chi(t,\vec{r},l)$, such that the equation of motion for χ is

$$\ddot{\chi}-2(p-1)^2\frac{l^2}{t^2}\overset{\star\star}{\chi}-\left(\frac{t_0}{t}\right)^{2p}l^{\frac{-2}{p-1}}\nabla_r^2\chi+\frac{1}{t^2}\left(-\frac{9}{4}p^2+3p+\frac{27}{8}\right)\chi=0. \tag{26}$$

We propose the following Fourier's expansion for χ

$$\chi(t,\vec{r},l)=\frac{1}{(2\pi)^{3/2}}\int d^3k_r\int dk_l\left[a_{k_rk_l}e^{i(\vec{k}_r.\vec{r}+k_ll)}\xi_{k_rk_l}(t,l)+a^\dagger_{k_rk_l}e^{-i(\vec{k}_r.\vec{r}+k_ll)}\xi^*_{k_rk_l}(t,l)\right], \tag{27}$$

where the creation and annihilation operators $a_{k_rk_l}$ and $a^\dagger_{k_rk_l}$ describe the algebra

$$\left[a_{k_rk_l},a^\dagger_{k'_rk'_l}\right]=\delta^{(3)}(\vec{k}_r-\vec{k}'_r)\delta(k_l-k'_l),\qquad [a_{k_rk_l},a_{k'_rk'_l}]=\left[a^\dagger_{k_rk_l},a^\dagger_{k'_rk'_l}\right]=0.$$

The dynamics of the modes $\xi_{k_rk_l}(t,l)$ is given by

$$\ddot{\xi}_{k_rk_l}+l^{-\frac{2}{p-1}}\left(\frac{t_0}{t}\right)^{2p}k_r^2\xi_{k_rk_l}+2(p-1)^2\frac{l^2}{t^2}\left\{-\frac{\partial^2}{\partial l^2}-2ik_l\frac{\partial}{\partial l}-\frac{1}{2(p-1)^2l^2}\left(\frac{9}{4}p^2-3p-\frac{27}{8}\right)+k_l^2\right\}\xi_{k_rk_l}=0. \tag{28}$$

Using the transformation $\xi_{k_rk_l}=e^{-i\vec{k}_l.\vec{l}}\tilde{\xi}_{k_rk_l}$, we obtain the following equation of motion for $\tilde{\xi}_{k_rk_l}$

$$\ddot{\tilde{\xi}}_{k_rk_l}-2(p-1)^2\frac{l^2}{t^2}\overset{\star\star}{\tilde{\xi}}_{k_rk_r}+\left[l^{-\frac{2}{p-1}}\left(\frac{t_0}{t}\right)^{2p}k_r^2-\left(\frac{9}{4}p^2-3p-\frac{27}{8}\right)\frac{1}{t^2}\right]\tilde{\xi}_{k_rk_l}=0. \tag{29}$$

The problem with this equation is that it is not separable. However, as we shall see later, this equation can be worked in the limit $p\gg1$, which is relevant for inflation.

2.1. *Inflationary case: $p\gg1$*

In the limit $p\gg1$ Eq. (29) can be simplified to

$$\ddot{\tilde{\xi}}_{k_rk_l}-2(p-1)^2\frac{l^2}{t^2}\overset{\star\star}{\tilde{\xi}}_{k_rk_r}+\left[k_r^2\left(\frac{t_0}{t}\right)^{2p}-\left(\frac{9}{4}p^2-3p-\frac{27}{8}\right)\frac{1}{t^2}\right]\tilde{\xi}_{k_rk_l}=0, \tag{30}$$

which, once normalized, has the following solution:

$$\tilde{\xi}_{k_r k_l}(t,l) = A_1\left[B_1\left(\frac{l}{l_0}\right)^{\frac{1+(1+4s^2)^{1/2}}{2}} + B_2\left(\frac{l}{l_0}\right)^{\frac{1-(1+4s^2)^{1/2}}{2}}\right]\sqrt{\frac{t}{t_0}}\mathcal{H}^{(1)}_{\nu}\big[x(t)\big], \tag{31}$$

where $\nu = \frac{\sqrt{1+9p^2-12p-27/2+8(p-1)^2 s^2}}{2(p-1)}$, $x(t) = \frac{k_r t_0^p}{t^{p-1}(p-1)}$ and s^2 is a separation constant.

Furthermore, the only solution of (23) that complies with the expression (21), is $\varphi_{(b)} = C$, where C is a constant. The equation of motion for the inflaton field (24), for $p \gg 1$, can be approximated to

$$\ddot{\delta\varphi} + \frac{(3p+1)}{t}\dot{\delta\varphi} - (p-1)^2\frac{l^2}{t^2}\overset{\star\star}{\delta\varphi} - (p-1)(4p-1)\frac{l}{t^2}\overset{\star}{\delta\varphi} - \left(\frac{t_0}{t}\right)^{2p}\nabla_r^2\delta\varphi = 0. \tag{32}$$

We can make the transformation $\delta\varphi(t,\vec{r},l) = (\frac{t}{t_0})^{\frac{-(3p+1)}{2}}(\frac{l}{l_0})^{\frac{-(4p-1)}{2(p-1)}}\Pi(t,\vec{r},l)$, so that the equation of motion for Π in the limit $p \gg 1$, is

$$\ddot{\Pi} - (p-1)^2\frac{l^2}{t^2}\overset{\star\star}{\Pi} + \frac{p}{2t^2}\left(1-\frac{p}{2}\right)\Pi - \left(\frac{t_0}{t}\right)^{2p}\nabla_r^2\Pi = 0. \tag{33}$$

The field Π can be expressed as a Fourier expansion in terms of its modes $\Pi_{k_r k_l}(t,\vec{r},l) = e^{i(\vec{k}_r\cdot+\vec{r}+k_l l)}\theta_{k_r k_l}(t,l)$, such that the dynamics for $\theta_{k_r k_l}$ is described by the equation

$$\ddot{\theta}_{k_r k_l} - (p-1)^2\frac{l^2}{t^2}\overset{\star\star}{\theta}_{k_r k_l} - 2ik_l(p-1)^2\frac{l^2}{t^2}\overset{\star}{\theta}_{k_r k_l} + \left[(p-1)^2\frac{l^2}{t^2}k_l^2 + \frac{p}{2t^2}\left(1-\frac{p}{2}\right) + \left(\frac{t_0}{t}\right)^{2p}k_r^2\right]\theta_{k_r k_l} = 0, \tag{34}$$

which has the following normalized solution

$$\theta_{k_r k_l} = F_1 e^{-ik_l l}\left[E_1\left(\frac{l}{l_0}\right)^{\frac{1}{2}+\frac{\sqrt{(p-1)^2+4q^2}}{2(p-1)}} + E_2\left(\frac{l}{l_0}\right)^{\frac{1}{2}-\frac{\sqrt{(p-1)^2+4q^2}}{2(p-1)}}\right]\sqrt{\frac{t}{t_0}}\mathcal{H}^{(1)}_{\mu}\big[x(t)\big]. \tag{35}$$

Here, $\mu = \frac{\sqrt{(p-1)^2+4q^2}}{2(p-1)} = \frac{\sqrt{1+4s^2}}{2}$, and $(F_1, E_1, E_2, q^2 = (p-1)^2 s^2)$ are constants.

3. Effective 4D dynamics

We consider the background metric (1). If we take a foliation such that $l = l_0$, the effective 4D background metric that results is

$$dS^2_{(b)} \to ds^2_{(b)} = l_0^2\,dt^2 - \left(\frac{t}{t_0}\right)^{2p} l_0^{\frac{2p}{p-1}}\,dr^2, \tag{36}$$

and the effective 4D Lagrangian is ${}^{(4)}\mathcal{L}(\varphi,\varphi_{,\mu}) = \frac{1}{2}g^{\mu\nu}\varphi_{,\mu}\varphi_{,\nu} - V(\varphi)$, such that the effective 4D background potential $V(\varphi_{(b)})$ induced on the metric (36) is

$$V(\varphi_{(b)}) = -\frac{1}{2}\bar{g}^{ll}\left[\varphi_{(b),l}\right]^2\Big|_{l=l_0} = \frac{1}{2}\frac{(p-1)^2}{t^2}\left[\varphi_{(b),l}\right]^2\Big|_{l=l_0}. \tag{37}$$

In the limit $p \gg 1$, which is relevant for inflation, the metric (36) can be approximated to

$$ds^2_{(b)} = l_0^2\,dt^2 - \left(\frac{t}{t_0}\right)^{2p} l_0^2\,dr^2. \tag{38}$$

The effective 4D potential (37) can be founded by solving the effective equation of motion for $\varphi_{(b)}$

$$\ddot{\varphi}_{(b)} + \frac{(3p+1)}{t}\dot{\varphi}_{(b)} - \frac{(p-1)^2}{t^2}l^2\overset{\star\star}{\varphi}_{(b)} - (p-1)(4p-1)\frac{l}{t^2}\overset{\star}{\varphi}_{(b)}\Big|_{l=l_0} = 0, \tag{39}$$

on the effective background metric (38). If we make $\varphi_{(b)}(t,l) = \varphi_1(t)\varphi_2(l)$, we obtain

$$\ddot{\varphi}_1 + \frac{(3p+1)}{t}\dot{\varphi}_1 = -\frac{m^2}{t^2}\varphi_1, \tag{40}$$

$$l^2(p-1)^2\overset{\star\star}{\varphi}_2 + (p-1)(4p-1)l\overset{\star}{\varphi}_2 = -m^2\varphi_2. \tag{41}$$

The general solutions for these equations are

$$\varphi_1(t) = t^{-3p/2}\left[B_1 t^{\frac{\sqrt{9p^2-4m^2}}{2}} + B_2 t^{-\frac{\sqrt{9p^2-4m^2}}{2}}\right], \tag{42}$$

$$\varphi_2(l) = l^{\frac{-3p}{2(p-1)}}\left[A_1 l^{\frac{\sqrt{9p^2-4m^2}}{2(p-1)}} + A_2 l^{-\frac{\sqrt{9p^2-4m^2}}{2(p-1)}}\right], \tag{43}$$

where A_1, A_2, B_1, B_2 are constants of integration and m is a separation constant. If we choose $A_1 = B_2 = m = 0$, we obtain that $\varphi_1(t) = B_1$ and $\varphi_2(l) = A_2 l^{3p/(p-1)}$, such that $\frac{\partial\varphi_{(b)}}{\partial t} = 0$ and $\frac{\partial\varphi_{(b)}}{\partial l} = \frac{-3p}{(p-1)} l^{-1}\varphi_{(b)}$. Therefore, the induced 4D background potential will be

$$V(\varphi_{(b)})|_{l=l_0} = \frac{9p^2}{2t^2l^2}\varphi_{(b)}^2\Big|_{l=l_0} = \frac{9p^2}{2t^2l_0^2}\varphi_{(b)}^2. \tag{44}$$

The metric (38), with the change of variables $\tau = l_0 t$ and $R = l_0 r$, becomes

$$ds_{(b)}^2 = d\tau^2 - \left(\frac{\tau}{\tau_0}\right)^{2p} dR^2. \tag{45}$$

Due to the fact $\frac{\partial\varphi_{(b)}}{\partial\tau} = 0$, the effective 4D background energy density is

$$\rho_b = \frac{1}{2}\left(\frac{\partial\varphi_{(b)}}{\partial\tau}\right)^2 + V(\varphi_{(b)})\Big|_{l=l_0} = \frac{9p^2}{2\tau^2}\varphi_{(b)}^2\Big|_{l=l_0} = \frac{3H^2(\tau)}{8\pi G}. \tag{46}$$

Here, the Hubble parameter $H(\tau) = \frac{1}{a}\frac{da}{d\tau} = p/\tau$, which is related to the scale factor $a(\tau) = a_0(\tau/\tau_0)^{2p}$ and $\varphi_{(b)}(\tau, l = l_0)$ is a constant: $\varphi_{(b)} = \frac{1}{2\sqrt{3\pi G}}$.

3.1. Effective 4D dynamics of metric fluctuations

Using the fact that the solution of Eq. (25), in the limital case $p \gg 1$, can be written as

$$\Psi(t,\vec{r},l) = \left[B_1\left(\frac{l}{l_0}\right)^{\frac{2p-5}{4(p-1)}+\frac{\sqrt{1+4s^2}}{2}} + B_2\left(\frac{l}{l_0}\right)^{\frac{2p-5}{4(p-1)}-\frac{\sqrt{1+4s^2}}{2}}\right]\psi(t,\vec{r}), \tag{47}$$

and that on the hypersurface $l = l_0$ one obtains

$$\Psi(t,\vec{r},l=l_0) = (B_1+B_2)\psi(t,\vec{r}), \tag{48}$$

$$\frac{\partial\Psi}{\partial l}\Big|_{l=l_0} = \left[\frac{2p-5}{4(p-1)}\frac{(B_1+B_2)}{l_0} + \frac{\sqrt{1+4s^2}}{2}\frac{(B_1-B_2)}{l_0}\right]\psi(t,\vec{r}), \tag{49}$$

$$\frac{\partial^2\Psi}{\partial l^2}\Big|_{l=l_0} = \left[\left(\frac{9}{16(p-1)^2} + s^2\right)\frac{(B_1+B_2)}{l_0^2} - \frac{3}{4}\frac{\sqrt{1+4s^2}}{(p-1)}\frac{(B_1-B_2)}{l_0^2}\right]\psi(t,\vec{r}). \tag{50}$$

Hence, the effective equation of motion for $\psi(\tau = l_0 t, \vec{R} = l_0\vec{r})$ on the hypersurface $l = l_0$ (which only is valid for $p \gg 1$), will be

$$\frac{\partial^2\psi}{\partial\tau^2} + \frac{3(p+2)}{\tau}\frac{\partial\psi}{\partial\tau} + \frac{\psi}{\tau^2}\left[\frac{21}{8} + \frac{21p}{2} - 2(p-1)^2s^2 + \frac{4p^2}{l_0^2}\right] - \left(\frac{\tau_0}{\tau}\right)^{2p}\nabla_R^2\psi = 0, \tag{51}$$

when we have used $B_1 = B_2$. Eq. (51) can be simplified using the transformation $\psi(\tau, \vec{R}, l_0) = (\frac{\tau}{\tau_0})^{\frac{-3(p+2)}{2}}\chi(\tau, \vec{R})$, such that the equation of motion for χ is

$$\frac{\partial^2\chi}{\partial\tau^2} + \frac{\chi}{\tau^2}\left[\frac{27}{2}p + \frac{69}{8} - 2(p-1)^2s^2 + \frac{4p^2}{l_0^2}\right] - \left(\frac{\tau_0}{\tau}\right)^{2p}\nabla_R^2\chi = 0. \tag{52}$$

The field $\chi(\tau, \vec{R})$ can be expanded in terms of their modes $\chi_{k_R}(\tau, \vec{R}) = e^{i\vec{k}_R\cdot\vec{R}}\tilde{\xi}_{k_R}(\tau)$, and the equation of motion for the τ-dependent modes $\tilde{\xi}_{k_R}(\tau)$, is

$$\frac{\partial^2\tilde{\xi}_{k_R}}{\partial\tau^2} + \left[k_R^2\left(\frac{\tau_0}{\tau}\right)^{2p} - \frac{[2(p-1)^2s^2 - \frac{4p^2}{l_0^2} - (\frac{27p}{2} + \frac{69}{8})]}{\tau^2}\right]\tilde{\xi}_{k_R} = 0. \tag{53}$$

The normalized solution for this equation is

$$\tilde{\xi}_{k_R}(\tau,\vec{R}) = A\sqrt{\frac{\tau}{\tau_0}}\mathcal{H}_\nu^{(1)}\big[x(\tau)\big], \tag{54}$$

where $\frac{4(p-1)}{\tau_0\pi}|A|^2 = 1$, $\mathcal{H}_\nu[x(\tau)]$ is the first kind Hankel function with argument $x(\tau) = k_R\frac{\tau_0^p}{(p-1)\tau^{p-1}}$ and $\nu = \frac{\sqrt{8(p-1)^2s^2-\frac{16p^2}{l_0^2}-54p-\frac{67}{2}}}{2(p-1)}$.

3.2. Energy density fluctuations

The energy density fluctuations on the effective 4D FRW metric is [14]

$$\frac{\delta\rho}{\langle\rho\rangle}\bigg|_{\mathrm{IR}} \simeq 2\Psi, \tag{55}$$

where the brackets $\langle\cdots\rangle$ denote the expectation value on the 3D hypersurface $\vec{R}(X,Y,Z)$. This approximation is valid during inflation on super Hubble scales. The amplitude for the 4D gauge-invariant metric fluctuations on cosmological scales is

$$\langle\Psi^2\rangle\big|_{\mathrm{IR}} \simeq 4B_1^2\langle\psi^2\rangle = 4B_1^2\left(\frac{\tau}{\tau_0}\right)^{-3(p+2)}\langle\chi^2\rangle\bigg|_{\mathrm{IR}}, \tag{56}$$

so that

$$\langle\Psi^2\rangle\big|_{\mathrm{IR}} \simeq \frac{4B_1^2(\frac{\tau}{\tau_0})^{-3(p+2)}}{(2\pi)^3}\int_0^{\epsilon(p/\tau)(\tau/\tau_0)^p} d^3k_R\,\tilde{\xi}_{k_R}\tilde{\xi}_{k_R}^* = \frac{1}{(2\pi)^3}\int_0^{\epsilon(p/\tau)(\tau/\tau_0)^p}\frac{dk_R}{k_R}\mathcal{P}_\Psi(k_R), \tag{57}$$

being $\epsilon \simeq 10^{-3}$ a dimensionless constant and $\mathcal{P}_\Psi(k_R) \sim k_R^{3-2\nu}$ the power spectrum of $\langle\Psi^2\rangle$. It is known from experimental data [16], that the spectral n_s index for this spectrum is

$$n_s = 0.97 \pm 0.03, \tag{58}$$

where, in our case, $n_s = 4 - \sqrt{8(p-1)^2s^2 - \frac{16p^2}{l_0^2} - 54p - \frac{67}{2}}/(p-1)|_{p\gg 1} \simeq 4 - \sqrt{8(s^2 - 2/l_0^2)}$. Hence, from the condition (58), we obtain

$$\frac{9}{8} + \frac{2}{l_0^2} < s^2 < \frac{(3.06)^2}{8} + \frac{2}{l_0^2}, \tag{59}$$

which provide us a cut for the separation constant s in terms of the value of the fifth coordinate l_0. Note that for each possible foliation $l_0^{(i)}$ there is a range of possible separation constants $s^{(i)}$.

3.3. Effective 4D dynamics of the inflaton field fluctuations

If we take into account that $\tau = l_0 t$ and $R = l_0 r$, the 5D equation (24) for $p \gg 1$, holds (see Eq. (32))

$$\frac{\partial^2\delta\varphi}{\partial\tau^2} + \frac{(3p+1)}{\tau}\frac{\partial\delta\varphi}{\partial\tau} - (p-1)^2\frac{l^2}{\tau^2}\frac{\partial^2\delta\varphi}{\partial l^2} - (p-1)(4p-1)\frac{l}{\tau^2}\frac{\partial\delta\varphi}{\partial l} - \left(\frac{\tau_0}{\tau}\right)^{2p}\nabla_R^2\delta\varphi = 0. \tag{60}$$

Note that for this limital case $\delta\varphi$ is independent of Ψ. Using the fact that

$$\delta\varphi|_{l=l_0} = (E_1+E_2)\mathcal{G}(\tau,\vec{R}), \tag{61}$$

$$\frac{\partial\delta\varphi}{\partial l}\bigg|_{l=l_0} = \frac{1}{2l_0(p-1)}\Big[-3p(E_1+E_2) + \sqrt{(p-1)^2+4q^2}(E_1-E_2)\Big]\mathcal{G}(\tau,\vec{R}), \tag{62}$$

$$\frac{\partial^2\delta\varphi}{\partial l^2}\bigg|_{l=l_0} = \frac{1}{4l_0^2(p-1)^2}\Big\{\big[(4p-1)^2+4q^2\big](E_1+E_2) + (2-8p)\sqrt{(p-1)^2+4q^2}(E_1-E_2)\Big\}\mathcal{G}(\tau,\vec{R}), \tag{63}$$

we obtain, for $E_1 = E_2$, the equation of motion for the function $\mathcal{G}$

$$\frac{\partial^2\mathcal{G}}{\partial\tau^2} + \frac{(3p+1)}{\tau}\frac{\partial\mathcal{G}}{\partial\tau} + \frac{\mathcal{G}}{\tau^2}\left[2p^2 + \frac{p}{2} - \left(\frac{1}{4}+q^2\right)\right] - \left(\frac{t_0}{t}\right)^{2p}\nabla_R^2\mathcal{G} = 0. \tag{64}$$

Now we can make the following transformation: $\mathcal{G}(\tau,\vec{R}) = (\frac{\tau}{\tau_0})^{\frac{-(3p+1)}{2}}\vartheta(\tau,\vec{R})$. Hence, the equation of motion for the τ-dependent modes $\tilde{\vartheta}_{k_R}(\tau)$ of $\vartheta(\tau,\vec{R})$, holds

$$\frac{\partial^2\tilde{\vartheta}_{k_R}}{\partial\tau^2} + \left[\left(\frac{\tau_0}{\tau}\right)^{2p} k_R^2 - \frac{(\frac{p^2}{4}+q^2-\frac{p}{2})}{\tau^2}\right]\tilde{\vartheta}_{k_R} = 0. \tag{65}$$

The normalized solution for the time dependent modes $\vartheta_{k_R}(\tau)$ is

$$\vartheta_{k_R}(\tau) = K_1\sqrt{\frac{\tau}{\tau_0}}\mathcal{H}_\sigma^{(1)}\big[x(\tau)\big], \tag{66}$$

where $\sigma = \frac{\sqrt{(p-1)^2+4q^2}}{2(p-1)} = \frac{\sqrt{1+4s^2}}{2}$, $x(\tau) = \frac{k_R\tau_0^p}{(p-1)\tau^{p-1}}$ and $\frac{4(p-1)}{\tau_0\pi}|K_1|^2 = 1$.

The squared $\delta\varphi$-fluctuations are

$$\big\langle\delta\varphi^2\big\rangle\big|_{\mathrm{IR}} \simeq \frac{4E_1^2}{(2\pi)^3}\left(\frac{\tau}{\tau_0}\right)^{-(3p+1)} \int_0^{\epsilon(p/\tau)(\tau/\tau_0)^p} d^3k_R\,\vartheta_{k_R}(\tau)\vartheta_{k_R}^*(\tau) = \frac{1}{(2\pi)^3}\int_0^{\epsilon(p/\tau)(\tau/\tau_0)^p} \mathcal{P}_{\delta\varphi}(k_R), \tag{67}$$

where $\mathcal{P}_{\delta\varphi} \sim k_R^{3-2\sigma}$ is the power-spectrum of $\langle\delta\varphi^2\rangle|_{\mathrm{IR}}$. Taking into account the condition (59), we obtain

$$\frac{\sqrt{1+\frac{9}{2}+\frac{8}{l_0^2}}}{2} < \sigma < \frac{\sqrt{1+\frac{(3.06)^2}{2}+\frac{8}{l_0^2}}}{2}, \tag{68}$$

such that, for a scale invariant power-spectrum $\mathcal{P}_{\delta\varphi}$ (i.e., for $\sigma = 3/2$), we obtain

$$2.2857 < l_0^2 < 2.4109, \tag{69}$$

which is a good estimation of l_0 for a nearly scale invariant $\mathcal{P}_{\delta\varphi}$ obtained from the experimental data (58).

4. Final Comments

In this Letter we have studied 4D gauge-invariant (scalar) metric fluctuations in space–time matter inflation, using the 5D Ponce de León background metric. This metric describes a 5D universe in an apparent vacuum $\bar{G}_{AB} = 0$, but however it is not Riemann flat. In general (i.e., for an arbitrary power of expansion p), the equations of motion for $\delta\varphi$ and Ψ are not factorizable, so that its treatment is very difficult. However, in the limital case $p \gg 1$ the treatment is possible because both, $\delta\varphi(t,\vec{r},l)$ and $\Psi(t,\vec{r},l)$ can be written as product of functions of t, $\vec{r}$ and l (i.e., both functions are factorizable). In such a case we have found that Ψ and $\delta\varphi$ become independents (as in an effective 4D de Sitter expansion studied in [14]), because the effective 4D background field $\varphi_{(b)}$ becomes a constant of $\tau = l_0 t$. This is not surprising because the case $p \gg 1$ describes an effective 4D asymptotic de Sitter expansion. In particular, using experimental values of n_s (for the $\delta\rho/\rho$-spectrum), we have found that the fifth coordinate used for the foliation $l = l_0$ could take values close to $l_0 \simeq 1.55$ in a scale invariant $\mathcal{P}_{\delta\varphi}$-power spectrum.

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